A Hierarchy of Ambiguity Aversion*

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Abstract

In many decision situations, the probabilities of uncertain events are not known ("ambiguity"). Presently one of the most popular approaches for decision-making under ambiguity is Choquet expected utility theory where subjective beliefs are represented by non-additive set functions ("capacities"). A much debated question is how to characterize attitudes towards ambiguous uncertainty. In this paper, we propose a new conceptual framework which allows different levels of ambiguity aversion. An act maps states of nature to deterministic or random outcomes. Higher ambiguity aversion is characterized by a stronger preference for mixing between acts. We show that the popular notions of ambiguity aversion by Schmeidler (1989) and by Ghirardato and Marinacci (2002) are extreme cases of our larger framework which we refer to as *hierarchy of ambiguity aversion.* We provide an axiomatization of the whole hierarchy; a by-product being the axiomatization of Choquet expected utility theory with exact capacities.

Keywords: Ambiguity aversion, Choquet expected utility, Exact capacities

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1 Introduction

Decision-makers often deal with situations where they do not know probabilities of uncertain events. In some situations, individuals behave as if they assigned subjective probabilities to the relevant events and then use a probabilistic decision criterion. This criterion can be expected utility as in Savage (1954) ("subjective expected utility theory") or, more generally, a nonexpected utility criterion as in Machina and Schmeidler (1992) ("probabilistic sophistication"). However, Ellsberg (1961) showed that individuals often do not behave as if they used subjective probabilities.

Since Knight (1921), we distinguish between measurable uncertainty ("risk"), which can be represented by probabilities, and unmeasurable uncertainty ("Knightian uncertainty", "ambiguity"), which cannot. Nevertheless, it took until the 1980s before models were developed that account for decision-making under ambiguity. One prominent approach is Choquet expected utility (CEU) theory introduced by Schmeidler (1989). In the CEU model, subjective beliefs are represented by a non-additive measure called *capacity*.

What is ambiguity aversion? This question has been discussed extensively in the literature and several definitions have been proposed.

Schmeidler (1989) defines ambiguity aversion via his ambiguity aversion axiom.[1] A different approach is taken by Epstein (1999) and by Ghirardato and Marinacci (2002). In the spirit of Yaari (1969), they define ambiguity aversion by justifying a benchmark for ambiguity neutrality as well as a comparative notion of ambiguity aversion. Combined this allows an absolute notion of ambiguity aversion. The approaches differ in their benchmarks for ambiguity neutrality. While Epstein uses probabilistic sophistication, Ghirardato and Marinacci use subjective expected utility.

This paper offers another approach for defining ambiguity aversion. We characterize the degree to which a decision maker (DM) wants to mix between acts and suggest that this degree reflects her ambiguity aversion: The stronger the preference for mixing, the higher is the DM's ambiguity aversion.

[1] Schmeidler calls the axiom "uncertainty aversion". In this paper, we use the term "uncertainty" in its generic sense, comprising risk and ambiguity. Therefore, we deviate from Schmeidler's terminology.

Throughout the paper the framework of Anscombe and Aumann (1963) (AA) is used. *Acts* are mappings from states into lotteries with known probabilities. Furthermore, we consider DM's with CEU preferences. We thus assume that the following axioms from Schmeidler (1989) hold for the preference relations discussed: Weak order, Comonotonic Independence, Continuity, Monotonicity, and Nondegeneracy. We refer to these axioms as standard CEU axioms.

The idea that a preference for mixing reflects ambiguity aversion is not new. Schmeidler (1989) suggests that an ambiguity averse DM always prefers a mixture between two acts over the least preferred of the two acts. The intuition behind this definition is that an ambiguity averse DM is always willing to trade "objective" uncertainty against ambiguous uncertainty.

This requirement that there is *always* a preference for mixtures is quite strong. Consider the example below: There are three states of nature s_1, s_2, s_3 and three acts f, g, h. Suppose that the DM is indifferent between the three acts. According to Schmeidler's ambiguity aversion axiom, an ambiguity averse DM (weakly) prefers **any** mixture between two of these acts to the acts themselves.

Example 1.

	s_1	s_2	s_3
f	0	1	2
g	2	1	0
h	4	0	1

It is plausible that an ambiguity averse DM prefers a 1/2-mix between f and g because this mix eliminates all ambiguous uncertainty. In other words, the 1/2-mix between f and g provides a perfect hedge against ambiguous uncertainty.[2] However, it is not clear why such a DM should prefer any of the other mixes. Weakly ambiguity averse individuals

[2] Or, more precisely, a perfect "probabilistic hedge" since the outcomes in the AA framework are expected utilities from "roulette" lotteries with known probabilities.

might only want to mix between acts if they obtain an act that leads to the same expected outcome in each state of nature.

A more ambiguity averse DM might not only prefer to mix whenever she obtains an act with only one expected outcome but also to mix whenever she obtains an act with only two expected outcomes (*binary act*). In the example above, this DM would also prefer a 1/2-mix between g and h. Still, it is not clear why she should prefer any of the other mixes.

In our view, the different degrees to which individuals want to mix between acts reveal different levels of ambiguity aversion. In this paper, we show that each degree of preference for mixing axiomatically characterizes a specific class of capacities in the CEU model. This yields a structure which we interpret as hierarchy of ambiguity aversion.

Schmeidler (1989) shows that his ambiguity aversion axiom characterizes CEU preferences with *convex capacities*[3], which lead to a very pessimistic decision criterion.

In the CEU model, the definition of ambiguity aversion by Ghirardato and Marinacci (2002) is represented by capacities that are pointwise dominated by at least one prior, called *balanced capacities*. Chateauneuf and Tallon (2002) axiomatically characterize balanced capacities. Their axiom postulates a preference for mixing between acts only if the mix eliminates all ambiguous uncertainty. As argued earlier, this is a much weaker requirement than the ambiguity aversion axiom.

Henceforth, we refer to the definition of Schmeidler (1989) as "S-Ambiguity Aversion" and to the one of Ghirardato and Marinacci (2002) as "GM-Ambiguity Aversion". We illustrate how the axioms that are relevant for these two definitions are related and that they are the extreme cases of a larger structure – the hierarchy of ambiguity aversion. Balanced capacities represent the weakest and convex capacities represent the strongest ambiguity aversion in this hierarchy.

In between these there are other capacity subclasses representing in-between levels of ambiguity aversion with the important class of exact capacities introduced by Schmeidler (1972) representing one of them. It is shown that exact capacities are axiomatically

[3]A capacity ν is *convex* if for any two events $A, B \subseteq S$, $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$.

characterized by a preference for mixing whenever the mix is a binary act.

This paper is organized as follows. After introducing the framework and notations in Section 2, we show in Section 3 how GM-Ambiguity Aversion and S-Ambiguity aversion are related and illustrate the hierarchy of ambiguity aversion. Section 4 introduces the concept of k - core capacities and presents our main mathematical result: The axiomatization of these capacity classes with exact capacities being one of them. Section 5 concludes with a summary. All proofs are in the appendix.

2 Preliminaries

Let S be a non-empty finite state space and Z be a set of consequences. We assume that Z is a 1 - dimensional vector space (for instance the real numbers). We denote by Σ the power set of S and by Y the set of distributions over Z with finite supports. Elements of S are called events and elements of Y are called lotteries.

Acts are functions from S to Y. The set of acts is denoted by $\mathcal{F}$. We abuse notation and denote by z the act that results in $z \in Z$ for all $s \in S$ and by y the act that results in $y \in Y$ for all $s \in S$. Thus Y also denotes the set of constant acts in $\mathcal{F}$ and Z a special subclass of this set (the set of constant acts which yield a degenerate lottery). Mixtures of acts are performed pointwise: For $f, g, \in \mathcal{F}$ and $\lambda \in [0,1]$ we as usual denote by $\lambda f + (1-\lambda)g$ the act which results in $\lambda f(s) + (1-\lambda)g(s) \in Y$ for all $s \in S$.

Let $\Delta(S)$ be the set of probability distributions over S. For a set function $\lambda : \Sigma \to [0,1]$ the sets $H_\lambda(E) = \{P \in \Delta(S) | P(E) = \lambda(E)\}$ and $\mathcal{H}_\lambda(E) = \{P \in \Delta(S) | P(E) \geq \lambda(E)\}$ are the hyperplane and upper-halfplane induced by $\lambda(E)$. The core of a set function λ is the set of probability distributions over S that pointwise dominate λ:

$$\begin{aligned} Core(\lambda) &= \{P \in \Delta(S) | P(E) \geq \lambda(E) \ \forall E \in \Sigma\} \\ &= \bigcap_{E \in \Sigma} \mathcal{H}_\lambda(E). \end{aligned}$$

For $n \in \{1, \ldots, |S|\}$, an act f is called a n - *act* if it maps to at most n different lotterie,s formally $|\{y \in Y | \exists s \in S : f(s) = y\}| \leq n$. We denote by $\mathcal{F}_n$ the set of n - acts.

Thus $\mathcal{F}_1 = Y$ and $\mathcal{F}_2$ is the set of binary acts. Every $f \in \mathcal{F}_2$ can be written as $f = x_E y$ for some $E \subseteq S$ and some $x, y \in Y$, which means that f results in x on E and in y on E^c – the complement of E. Of course: $\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{|S|} = \mathcal{F}$.

Let u be a utility function that assigns utility $u(z)$ to each consequence $z \in Z$. We slightly abuse notation and denote by u also the expected utility function that assigns expected utility $u(y)$ to each lottery $y \in Y$. An act is called *n-expected utility act* if it maps to at most n different expected outcomes, formally $|\{a \in \mathbb{R} | \exists s \in S : u(f(s)) = a\}| \leq n$. The set of n-expected utility acts is denoted by $\mathcal{F}_n^u$. Note that for every utility function u it holds that $\mathcal{F}_n \subseteq \mathcal{F}_n^u$ but the reverse does not hold.

Preferences over $\mathcal{F}$ are represented by a binary relation $\succsim$, where $\succ$ and $\sim$ respectively denote the asymmetric and symmetric components of $\succsim$. Throughout the paper it is assumed that the standard axioms of CEU theory hold for the preference relations discussed. Hence, for every preference relation $\succsim$ there exists a unique normalized monotone set function (*capacity*) $\nu : \Sigma \to [0, 1]$, as well as an affine utility function $u : Z \to \mathbb{R}$ such that

$$f \succsim g \text{ if and only if } \int_S u(f(s))\, d\nu \geq \int_S u(g(s))\, d\nu,$$

where the integral is a Choquet integral (cf. Choquet, 1954) taken with respect to the capacity ν. We define $H_\nu^u(f) = \{P \in \Delta(S) | \int_S u(f(s))\, dP = \int_S u(f(s))\, d\nu\}$. For a binary act $f = x_E y$ with $x \succsim y$ and $\emptyset \neq E \neq S$ this implies $H_\nu^u(f) = H_\nu(E)$ for all utility functions u.

There are three capacity classes that are particularly relevant for this paper: balanced, exact and convex capacities. A capacity is *balanced* if its core is nonempty. A capacity is *exact* if $\nu(E) = \min_{P \in Core(\nu)} P(E)$ for all $E \in \Sigma$. Note that this is equivalent to $H_\nu(E) \cap Core(\nu) \neq \emptyset$ for all $E \in \Sigma$. A capacity ν is *convex* if $\nu(E_1) + \nu(E_2) \leq \nu(E_1 \cup E_2) + \nu(E_1 \cap E_2)$ for all $E_1, E_2 \in \Sigma$. It is a well-known fact that convex capacities are exact and exact capacities are balanced but the reverse directions do not hold.

3 The Hierarchy of Ambiguity Aversion

3.1 S-Ambiguity Aversion and GM-Ambiguity Aversion

Under the standard CEU axioms, the following two axioms characterize the definitions of ambiguity aversion of Schmeidler (1989) and of Ghirardato and Marinacci (2002), respectively.[4] We use the here stated alternative versions since this allows to put them into perspective.[5] Consider the following two axioms.

Axiom 1 (*S* - Ambiguity Aversion)**.** $f_1, \ldots, f_n \in \mathcal{F}$, $\alpha_1, \ldots, \alpha_n \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$, $\sum_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}$ such that $f_1 \sim \cdots \sim f_n$ implies $f \succsim f_1$.

Axiom 2 (*CTGM* - Ambiguity Aversion)**.** $f_1, \ldots, f_n \in \mathcal{F}$, $\alpha_1, \ldots, \alpha_n \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$, $\sum_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}_1$ such that $f_1 \sim \cdots \sim f_n$ implies $f \succsim f_1$.

The difference between the two axioms might seem small but is crucial for this paper. Axiom 1 requires a preference for mixing between indifferent acts, independent of what kind of act the mixture is. A DM who satisfies this axiom always has a preference to trade objective mixing for subjective mixing. Axiom 1 holds if and only if the capacity is convex.

Axiom 2 requires preference for mixing between indifferent acts if the mixture constitutes a constant act, which is a weaker requirement. A DM who satisfies this axiom wants to mix amongst acts if this mix eliminates all ambiguity, leaving only risk. Axiom 2 holds if and only if the capacity is balanced.

Both these definitions have compelling justifications and we do not aim to side with either of them. Rather, we illustrate in the following that these two approaches are the extreme cases of what we refer to as a "Hierarchy of Ambiguity Aversion". Axiom 2 represents the weakest level of ambiguity aversion. Axiom 1 represents the strongest level of ambiguity aversion.

[4] Axiom 2 originates from Chateauneuf and Tallon (2002), which explains the name of the Axiom.

[5] Axiom 1 is equivalent to the Uncertainty Aversion Axiom in Schmeidler (1989). Axiom 2 is equivalent to the Sure "Expected" Utility Diversification Axiom in Chateauneuf and Tallon (2002). We prove these claims in the Appendix. Most importantly we show that we can drop the "expected" utility part in the axiom of Chateauneuf and Tallon (2002). Thus it is sufficient to consider acts $\mathcal{F}_1$ instead of constant expectation acts $\mathcal{F}_1^u$. Consequently the here stated version can be stated purely in terms of preferences.

In between these extremes are other levels, one of them being characterized by our Axiom "Binary Ambiguity Aversion" that induces the class of exact capacities. Each of these levels is characterized by a different degree of preference for mixing amongst acts. The further we go up the hierarchy, the more pronounced is the preference for mixing, i.e. the DM has a greater preference for trading objective for subjective mixing. We show that each of these levels is represented by a distinct class of capacities that we refer to as k - core capacities. We thus axiomatize the whole hierarchy, including exact capacities.

3.2 The Hierarchy: k-Ambiguity Aversion

Consider the following axiom.

Axiom 3 (Binary Ambiguity Aversion). $f_1, \ldots, f_n \in \mathcal{F}$, $\alpha_1, \ldots, \alpha_n \geq 0$, $\sum\limits_{i=1}^{n} \alpha_i = 1$, $\sum\limits_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}_2$ such that $f_1 \sim \cdots \sim f_n$ implies $f \succsim f_1$.

Axiom 3 requires a preference for mixing if the mixture constitutes a binary act. A DM satisfying this axiom wants to mix amongst acts if this mix reduces ambiguity to at most two different outcomes. It is therefore weaker than Axiom 1 and stronger than Axiom 2. As we show in the Main Theorem, Axiom 3 characterizes exact capacities which so far have lacked an axiomatization. The logic of the axioms introduced so far suggest the following class of axioms that we refer to as k - Ambiguity Aversion, where $1 \leq k \leq |S|$.

Axiom 4 (k - Ambiguity Aversion). $f_1, \ldots, f_n \in \mathcal{F}$, $\alpha_1, \ldots, \alpha_n \geq 0$, $\sum\limits_{i=1}^{n} \alpha_i = 1$, $\sum\limits_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}_k$ such that $f_1 \sim \cdots \sim f_n$ implies $f \succsim f_1$.

Remark. It is straightforward that for $k = 1$, we obtain $CTGM$ - Ambiguity Aversion (Axiom 2), which characterizes balanced capacities. The case $k = |S|$ is S - Ambiguity Aversion (Axiom 1) and therefore characterizes convex capacities. Furthermore, $2-$ambiguity aversion is Binary Ambiguity Aversion (Axiom 3), which yields exact capacities as the Main Theorem shows.

The strength of Axiom 4 increases with k. As k becomes larger the preference for mixing amongst acts increases. We thus suggest that a larger k corresponds to greater ambiguity aversion.

4 Main Theorem

4.1 k - Core Capacities

In the Main Theorem we show that, for $k \in \{1, \dots, |S|\}$, the axiom k - Ambiguity Aversion characterizes a specific class of capacities that we refer to as **k - core capacities**.

Definition 1 (k - core capacity)**.** Consider $k \in \{1, \dots, |S|\}$ and an affine utility function u. A capacity ν has a k - *core* if $Core(\nu) \neq \emptyset$ and for all $f \in \mathcal{F}_k$

$$H_\nu^u(f) \cap Core(\nu) \neq \emptyset. \tag{1}$$

The case $k = 1$ corresponds to balanced capacities since for constant acts (1) trivially holds. The case $k = 2$ corresponds to exact capacities since for a binary act $f = x_E y$ with $x \succ y$ it holds that $H_\nu^u(f) = H_\nu(E)$. The case $k = |S|$ corresponds to convex capacities due to the result of Schmeidler (1989) that a capacity is convex if and only if $\int_S u(f(s))\, d\nu = \min_{P \in Core(\nu)} \int_S u(f(s))\, dP$ for all $f \in \mathcal{F}$, which implies (1).

The following examples illustrate the concept of k - core capacities. For simplicity we assume that consequences are in terms of expected utilities throughout the examples. Hence, an act is a vector $f = \big(u(f(s_1), \dots, u(f(s_{|S|})\big)$. For the sake of simplicity, the Choquet integral of an act f with respect to a capacity is denoted by $\int f d\nu$.

Example 2. *Consider the state space $S = \{s_1, s_2, s_3\}$ and the capacity ν, its core is illustrated in Figure 1:*

$$\nu(E) = \begin{cases} 1 & ,\textit{for } E = S \\ 0 & ,\textit{for } E = \emptyset \\ \frac{1}{6} & ,\textit{for } |E| = 1 \\ \frac{1}{3} & ,\textit{for } E = \{s_1, s_2\}; E = \{s_2, s_3\} \\ \frac{1}{4} & ,\textit{for } E = \{s_1, s_3\} \end{cases}$$

The capacity has a 1 - core, but not a 2 - core, which means that it is balanced but *not exact.*[6] *The reason for the latter is that for the binary act* $f = 1_{\{s_1,s_3\}}0$ *we have* $H_\nu(f) \cap Core(\nu) = \emptyset$ *or equivalently* $\nu\big(\{s_1, s_3\}\big) < \min\limits_{P \in Core(\nu)} P\big(\{s_1, s_3\}\big)$.

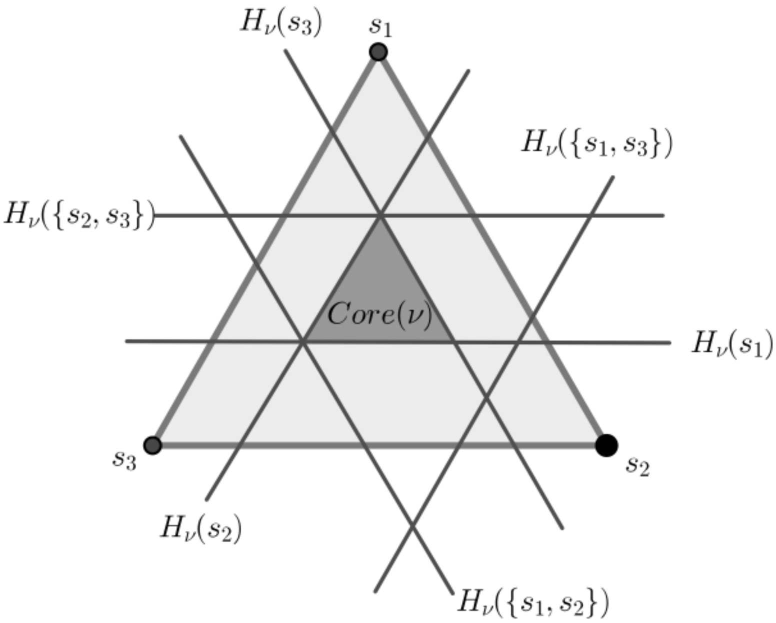

Figure 1: A capacity which has a 1 - core, but not a 2 - core

Example 3. *Consider the state space* $S = \{s_1, s_2, s_3, s_4\}$ *and the following capacity* μ*, the edges of its core illustrated in black in Figure 2:*

$$\mu(E) = \begin{cases} 1 & ,\textit{for } E = S \\ \frac{1}{10} & ,\textit{for } E = \{s_1, s_2\}; E = \{s_1, s_3\}; E = \{s_1, s_2, s_3\}; \\ & E = \{s_1, s_2, s_4\}; E = \{s_1, s_3, s_4\}; E = \{s_2, s_3, s_4\} \\ 0 & ,\textit{for } E \textit{ otherwise} \end{cases}$$

The capacity has a 2 - core, but not a 3 - core and therefore also not a 4 - core. That is, the capacity is exact but not convex. Exactness follows through Definition 1 from the fact that the distribution $P = (\frac{1}{10}, 0, 0, \frac{9}{10}) \in Core(\nu)$ *and* $P \in H_\nu(E)$ *for* $E \in \big\{\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_2, s_3\}\big\}$.[7] *Alternatively exactness is implied via the standard definition since* $\nu(E) \cap Core(\nu) \neq \emptyset$. *To see via Definition 1 that the capacity is not*

[6] This implies that it is not convex either.

[7] With the help of Figure 2 it can be checked that the hyperplanes of all other events touch $Core(\nu)$.

convex consider the tertiary act $f = 2_{\{s_1\}}1_{\{s_2,s_3\}}0 \in \mathcal{F}_3$. *The evaluation of* f *given* ν *if* $\int f\ d\nu = \frac{1}{10}$. *The set*

$$\begin{aligned} H_\mu(f) &= \{P \in \Delta(S) | \int f\ dP = \int f\ d\nu\} \\ &= \{P \in \Delta(S) | \int f\ dP = \frac{1}{10}\} \end{aligned}$$

has an empty intersection with $Core(\mu)$. *This can be seen from the picture in which the hyperplane* $H_\mu(f)$ *is illustrated.*

Alternatively non-convexity can be deduces via the standard definition from the fact that $\mu(\{s_1, s_2\}) + \mu(\{s_1, s_3\}) > \mu(\{s_1, s_2, s_3\}) + \mu(\{s_1\})$.

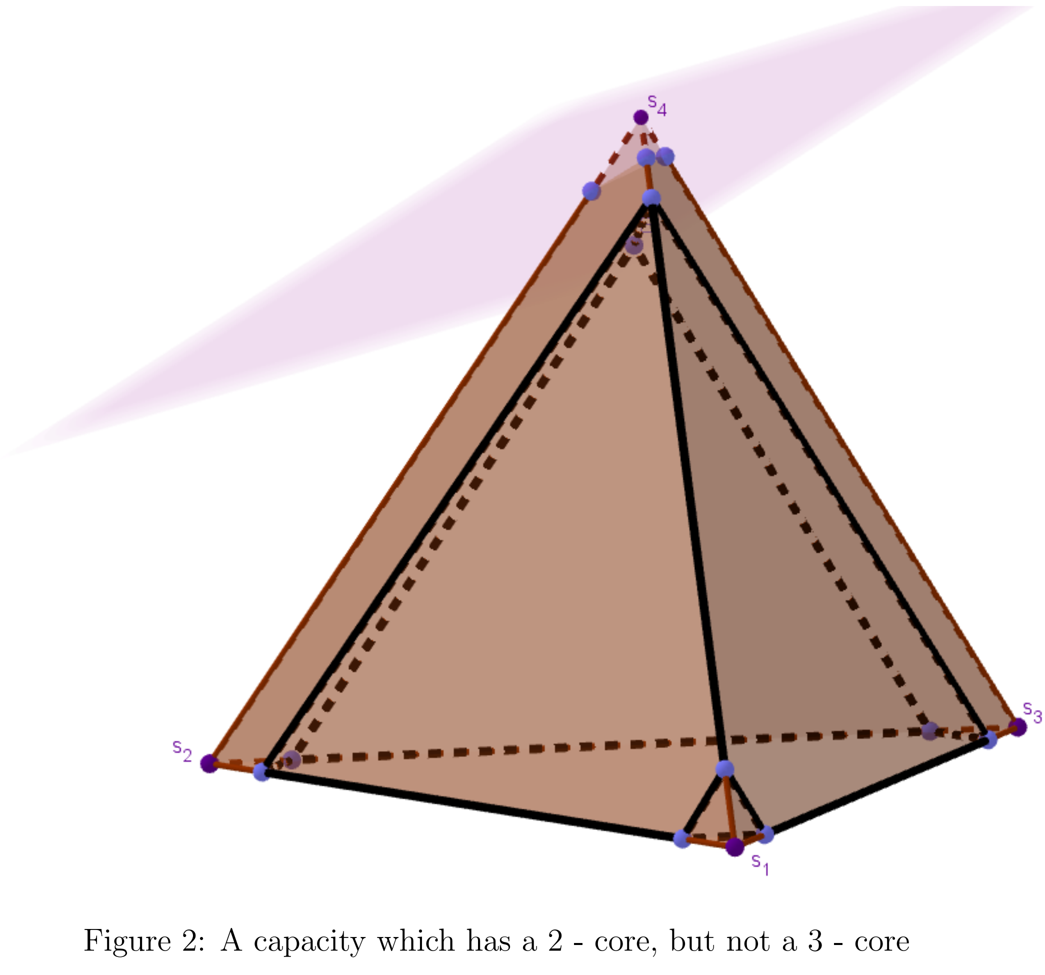

Figure 2: A capacity which has a 2 - core, but not a 3 - core

4.2 The Theorem: Axiomatization of k - Core Capacities

The following Lemma is the first step towards showing that the axiom k - Ambiguity Aversion characterizes k - core capacities. It shows that a capacity has a k - core if and only if all k - acts are "evaluated at the core".

Lemma 1. *Let ν be a balanced capacity, i.e. $Core(\nu) \neq \emptyset$. Then*

$$\int_S u(f(s))\, d\nu \;\leq\; \min_{P \in Core(\nu)} \int_S u(f(s))\, dP \text{ for all } f \in \mathcal{F}.$$

Furthermore, the following are equivalent:

(i) ν has a k - core.

(ii) $\int_S u(f(s))\, d\nu = \min_{P \in Core(\nu)} \int_S u(f(s))\, dP$ for all $f \in \mathcal{F}_k$.

Lemma 1 has very specific consequences on the preference for mixing between acts. To illustrate this consider a DM whose preferences can be represented by a balanced capacity. Assume that the DM is indifferent between the acts $f_1, \ldots, f_n$. Now consider some convex mixture f of those acts: $f = \sum_{i=1}^{n} \alpha_i f_i$. When does the DM prefer this mixture to the original acts, i.e. when does it hold that $f \succsim f_1$? What are the corresponding characteristics of the capacity ν? It turns out that $f \succsim f_1$ can be guaranteed to hold if and only if f is "evaluated at the core", i.e. when $H_\nu^u(f) \cap Core(\nu) \neq \emptyset$.

For the "if" part of this claim consider some $P \in H_\nu^u(f) \cap Core(\nu)$. The first part of Lemma 1 implies that $\int_S u(f_i(s)) d\nu \leq \int_S u(f_i(s))\, dP$ for all $\leq i \leq n$. Consequently, $\sum_{i=1}^{n} \alpha_i \int_S u(f_i(s))\, d\nu \leq \sum_{i=1}^{n} \alpha_i \int_S u(f_i(s))\, dP = \int_S u(f(s))\, dP$. This implies $f \succsim f_1$.

The "only if" part is less intuitive. Assume that f is not "evaluated at the core", which means that $\int_S u(f(s))\, d\nu < \min_{P \in Core(\nu)} \int_S u(f(s))\, dP$. It turns out that we can always find acts $f_1 \sim \cdots \sim f_n$ such that $\sum_{i=1}^{n} \alpha_i f_i = f$ and $f_1 \succ f$.[8] Thus an act which is not "evaluated at the core" is not necessarily preferred when constituting a mixture amongst indifferent acts.

[8]For a proof of this claim we refer the interested reader to the proof of of the Main Theorem in the Appendix.

The above demonstrates that k - core capacities are characterized by the k - Ambiguity Aversion axiom. The following theorem states this result formally.

Main Theorem. *Consider a preference relation $\succsim$ satisfying the standard CEU axioms with corresponding capacity ν. The following are equivalent:*

(i) ν has a k - core.

(ii) $\succsim$ satisfies the axiom k - Ambiguity Aversion.

We reconsider the previous examples in light of Lemma 1 and the Main Theorem.

Example 2 (continued)**.** *Reconsider the capacity ν. For the binary act $f = 1_{\{s_1,s_3\}}0$ we have*

$$\int f \; d\nu = \nu(\{s_1, s_3\}) = \frac{1}{4} < \frac{1}{3} = \int f \; dP^* = \min_{P \in Core(\nu)} \int f \; dP.$$

The binary act f is not evaluated at the core. Therefore by Lemma 1, ν does not have a 2 - core. The same conclusion can be made through the Main Theorem. For ν the 2 - Ambiguity Aversion Axiom fails, whereas the 1 - Ambiguity Aversion Axiom holds. To see that the 2 - Ambiguity Averse Axiom fails consider the acts $f_1 = 2_{\{s_1\}}0$ and $f_2 = 2_{\{s_3\}}0$. Obviously $f_1 \sim f_2$ and $\frac{1}{2}f_1 + \frac{1}{2}f_2 = f$. However

$$\int f \; d\nu = \frac{1}{4} < \frac{1}{3} = \int f_1 \; d\nu,$$

thus $f_1 \succ f$. Hence, a DM whose preferences are represented by ν does not prefer the 1/2-mix f to the acts f_1 and f_2.

Example 3 (continued)**.** *Reconsider the capacity μ. For the tertiary act $f = 2_{s_1}1_{\{s_2,s_3\}}0 \in \mathcal{F}_3$ it holds that $\int f \; d\mu < \min_{P \in Core(\mu)} \int f \; dP$, thus f is not evaluated at the core. This implies that μ does not have a 3 - core and therefore also not a 4 - core, i.e. μ is not convex.*

Again the same conclusion can be drawn from the Main Theorem. Consider the acts $f_1 = 2_{\{s_1,s_2\}}0, f_2 = 2_{\{s_1,s_3\}}0$. It holds that $f_1 \sim f_2$ as well as $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$. Furthermore

$$\int f \; d\mu = 2\mu(\{s_1\}) + \mu(\{s_1, s_2, s_3\}) = \frac{1}{10} < \frac{2}{10} = 2\mu(\{s_1, s_2\}) = \int f_1 \; d\mu.$$

The 3 - Ambiguity Aversion axiom therefore fails. This implies that μ is not convex.

5 Conclusion

This paper introduces the notion of k - Ambiguity Aversion: A higher k - level of ambiguity aversion is characterized by a stronger preference for mixing between acts. This leads to a structure which is refer to as hierarchy of ambiguity aversion. The extreme cases of this structure correspond to the popular definitions of ambiguity aversion of Schmeidler (1989) and of Ghirardato and Marinacci (2002). Our notion thus offers a unifying framework for these definitions.

We show that a Choquet expected utility preference relation satisfies k - Ambiguity Aversion if and only if it induces a specific capacity with a special core property – the k-core capacity. In particular, exact capacities introduced by Schmeidler (1972) are a special case of k - core capacities. We therefore axiomatically characterize this important subclass of capacities, which so far has lacked an axiomatization.

Further, it is shown that the lowest level of k - Ambiguity Aversion induces a capacity with a non-empty core. These capacities correspond to Ghirardato and Marinacci's definition of ambiguity aversion in the Choquet expected utility model. Choquet preferences satisfy the highest level of k - Ambiguity Aversion if and only if the corresponding capacity is convex, which corresponds to Schmeidler's definition.

Appendix

This appendix provides the proofs of the results presented in this paper. First we prove that Axiom 1 is indeed equivalent to Schmeidler's Uncertainty Aversion and that Axiom 2 is equivalent to the Sure "Expected" Utility Diversification assumption of Chateauneuf and Tallon (2002).

Axiom (Uncertainty Aversion)**.**
$f_1, f_2 \in \mathcal{F}$ with $f_1 \sim f_2$, $\alpha \in [0,1] \implies \alpha f_1 + (1-\alpha) f_2 \succeq f_1$.

Axiom (S - Ambiguity Aversion)**.**
$f_1 \sim \cdots \sim f_n, \sum\limits_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}, \alpha_1, \ldots, \alpha_n \geq 0, \sum\limits_{i=1}^{n} \alpha_i = 1 \implies f \succsim f_1$.

Axiom (Sure "Expected" Utility Diversification)**.**
$f_1 \sim \cdots \sim f_n, \sum\limits_{i=1}^{n} \alpha_i f_i = f$ such that $f(s) = f(s')$ for all $s, s' \in S, \alpha_1, \ldots, \alpha_n \geq 0, \sum\limits_{i=1}^{n} \alpha_i = 1 \implies f \succsim f_1$.

Axiom ($CTGM$ - Ambiguity Aversion)**.**
$f_1 \sim \cdots \sim f_n, \sum\limits_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}_1, \alpha_1, \ldots, \alpha_n \geq 0, \sum\limits_{i=1}^{n} \alpha_i = 1 \implies f \succsim f_1$.

Lemma 2. *Under the standard CEU axioms, Uncertainty Aversion is equivalent to S - Ambiguity Aversion.*

Proof. The nontrivial direction is that Uncertainty Aversion implies S - Ambiguity Aversion. We prove it by induction over n. The base case is $n = 2$ which holds by assumption. Assume that it holds for $n-1$. We need to show that it then also holds for n.

Assume that $f_1 \sim \cdots \sim f_n$ and $f = \sum\limits_{i=1}^{n} \alpha_i f_i = f \in \mathcal{F}$. Consider the act

$$f' = \frac{\sum\limits_{i=2}^{n} \alpha_i f_i}{1-\alpha_1}.$$

Note that indeed it holds that $\frac{\sum_{i=2}^{n} \alpha_i}{1-\alpha_1} = 1$. So the act f' is a convex mix of the $n-1$ acts $f_2, \ldots, f_n$. By the induction hypothesis we have $f' \succsim f_2$ which implies that $f' \succsim f_1$.

It holds that $f = \alpha_1 f_1 + (1-\alpha_1)f'$. Therefore since 2. holds we have that

$$f \succsim f_1.$$

This finishes the induction step. $\square$

Lemma 3. *Under the standard CEU axioms, Sure "Expected" Utility Diversification is equivalent to CTGM - Ambiguity Aversion.*

Proof. The nontrivial direction is that *CTGM* - Ambiguity Aversion implies Sure "Expected" Utility Diversification. Assume $f_1 \sim \cdots \sim f_n$ such that $\sum_{i=1}^{n} \alpha_i u(f_i(s_j)) = a \in \mathbb{R}$ for all $s_j \in S$. Since Z is a 1-dimensional vector space there exists an $x \in Z$ such that $u(x) = a$. Furthermore there exist $x_{ji} \in Z$ such that $u(f_i(s_j)) = u(x_{ji})$ for all $j \in \{\ldots |S|\}$ and $i \in \{1, \ldots n\}$. For $i \in \{1, \ldots n\}$ define $g_i \in \mathcal{F}$ such that $g_i(s_j) = x_{ji}$ for all $j \in \{\ldots |S|\}$. It holds that $g_1 \sim \cdots \sim g_n$ and $\sum_{i=1}^{n} \alpha_i u(g_i(s_j)) = u(x)$ and thus due to the affinity of u we have $\sum_{i=1}^{n} \alpha_i g_i(s_j) = x$. Therefore due to the axiom *CTGM* - Ambiguity Aversion it holds that $g_1 \succsim x$ implying $f_1 \succsim x$. $\square$

Recall that we assume a finite state space S with Σ being the powerset of S, i.e. the set of events. We denote by E^* the indicator function of the event E. Let $\mathbf{B}(\Sigma)$ be the vectorspace spanned by $\{E^* | E \in \Sigma\}$. Let $ba(\Sigma)$ be the set of bounded additive set functions on Σ. It is well-known that $ba(\Sigma)$ is isometrically isomorphic to the norm-dual of $B(\Sigma)$. For $\lambda \in ba(\Sigma)$ denote by λ^* the corresponding linear functional on $\mathbf{B}(\Sigma)$, so λ^* is in particular the linear functional such that $\lambda(E) = \lambda^*(E^*)$ for all $E \in \Sigma$. A chain is a tuple of events $(F_1, \ldots, F_k)$ such that $\emptyset = F_1 \subsetneq \cdots \subsetneq F_k \subsetneq S$. $\mathcal{C}$ denotes the set of all chains and $\mathcal{C}_k$ the set of chains of length at most k, $\mathcal{C}_{|S|}$ is the set of maximal chains. First we prove the following more general version of Lemma 1.

Lemma 4. *Let ν be a balanced capacity, i.e. $Core(\nu) \neq \emptyset$ and u an affine utility function. Then*

$$\int u(f)\ d\nu \leq \min_{P \in Core(\nu)} \int u(f)\ dP \text{ for all } f \in \mathcal{F}.$$

Furthermore the following are equivalent:

(i) ν *has a k - core*

(ii) $\int u(f)\, d\nu = \min_{P \in Core(\nu)} \int u(f)\, dP$ *for all* $f \in \mathcal{F}_k$.

(iii) *For all chains* $(F_1, \ldots, F_k) \in \mathcal{C}_k$

$$H_\nu(F_1) \cap \cdots \cap H_\nu(F_k) \cap Core(\nu) \neq \emptyset.$$

(iv) *For all chains* $(F_1, \ldots, F_k) \in \mathcal{C}_k$

$$\sum_{i=1}^{k} \nu(F_i) = \min_{P \in Core(\nu)} \sum_{i=1}^{k} P(F_i).$$

Proof. 1. Consider $f \in \mathcal{F}$. There exists a $k \in \{1, \ldots, |S|\}$ such that $f \in \mathcal{F}_k$. There exists a chain $(F_1, \ldots, F_k) \in \mathcal{C}_k$ and $x_1, \ldots, x_k \in Y$ with $x_1 \succsim x_2 \succsim \cdots \succsim x_k$ such that $f(s) = x_i$ for all $s \in F_{i+1} \backslash F_i, i \in \{1, \ldots, k-1\}$ and $f(s) = x_k$ for $s \in S \backslash F_k$. Without loss of generality we assume that $x_k \succ 0$ (if this is not the case add a sufficiently large constant to f). Define the following binary acts $f_i = (x_i - x_{i+1})_{F_{i+1} \backslash F_i} 0$ for $i \in \{1, \ldots, k-1\}$, $f_k = (x_k - x_{k-1})_{S \backslash F_k} 0$ and $f_{k+1} = x_k$. Then the acts $f, f_1, \ldots, f_k$ are pairwise comonotonic and $f = \sum_{i=1}^{k+1}$. Consider $P \in Core(\nu)$. We have that

$$\begin{aligned}
\int u(f)\ d\nu &= \int \sum_{i=1}^{k+1} u(f_i)\ d\nu \\
&= \sum_{i=1}^{k+1} \int u(f_i)\ d\nu \\
&= \sum_{i=1}^{k} u(x_i - x_{i-1})\nu(F_{i+1} \backslash F_i) + u(x_k - x_{k-1})\nu(S \backslash F_k) + u(x_k) \\
&\leq \sum_{i=1}^{k} u(x_i - x_{i-1})P(F_{i+1} \backslash F_i) + u(x_k - x_{k-1})P(S \backslash F_k) + u(x_k) \\
&= \sum_{i=1}^{k+1} \int u(f_i)\ dP \\
&= \int \sum_{i=1}^{k+1} u(f_i)\ dP \\
&= \int f\ dP,
\end{aligned}$$

the interchanging of the sum and the integral being possible due to comonotonic independence. Since $P \in Core(\nu)$ was arbitrary and $Core(\nu)$ is compact we have that

$$\int u(f)\ d\nu \leq \min_{P \in Core(\nu)} \int u(f)\ dP.$$

2. Note that the $k = 1$ case is trivial. For the remainder of the proof consider the case $k > 1$.

"$(i) \implies (ii)$": Due to the first part of the Lemma we have that $\int u(f)\ d\nu \leq \min_{P \in Core(\nu)} \int u(f)\ dP$. Consider an act $f \in \mathcal{F}_k$ and assume that $Q \in H_\nu^u(f) \cap Core(\nu)$. Then

$$\int u(f)\ d\nu = \int u(f)\ dQ \geq \min_{P \in Core(\nu)} \int u(f)\ dP.$$

Thus (ii) holds.

"$(ii) \implies (i)$": Assume ν does not have a k - core. Then there exists an $f \in \mathcal{F}_k$ such that $H^u_\nu(f) \cap Core(\nu) = \emptyset$. The first part of the lemma implies

$$\int u(f)\ d\nu < \min_{P \in Core(\nu)} \int u(f)\ dP.$$

Thus (ii) fails.

"$(ii) \implies (iii)$": For $f \in \mathcal{F}_k$ consider the construction of a chain $(F_1, \ldots, F_k)$ as in the proof of the first part of the lemma. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \min_{P \in Core(\nu)} \int u(f_i)\ dP &= \sum_{i=1}^{k+1} \int u(f_i)\ d\nu \\
&= \int u(f)\ d\nu \\
&= \min_{P \in Core(\nu)} \int u(f)\ dP \\
&= \min_{P \in Core(\nu)} \sum_{i=1}^{k+1} \int u(f_i)\ dP,
\end{aligned}$$

where the first equality holds since $k > 1$ and all the f_i are binary acts, the second inequality holds since all the considered acts are pairwise comonotonic and the third inequality holds since we assume (ii). Thus $H_\nu(F_1) \cap \cdots \cap H_\nu(F_k) \cap Core(\nu) \neq \emptyset$.

"$(iii) \implies (ii)$": Assume that $Q \in H_\nu(F_1) \cap \cdots \cap H_\nu(F_k) \cap Core(\nu)$. It follows that

$$\begin{aligned}
\min_{P \in Core(\nu)} \int u(f)\ dP &= \min_{P \in Core(\nu)} \sum_{i=1}^{k+1} \int u(f_i)\ dP \\
&= \sum_{i=1}^{k+1} \int u(f_i)\ dQ \\
&= \sum_{i=1}^{k+1} \int u(f_i)\ d\nu \\
&= \int u(f) d\nu.
\end{aligned}$$

Therefore (ii) holds.

"$(iii) \iff (iv)$": Assume that $Q \in H_\nu(F_1) \cap \cdots \cap H_\nu(F_k) \cap Core(\nu)$. This implies that $Q(F_i) = \nu(F_i)$ as well as $Q(F_i) = \min_{P \in Core(\nu)} P(F_i)$. This is equivalent to

$$\begin{aligned}\sum_{i=1}^{k} \nu(F_i) &= \sum_{i=1}^{k} \min_{P \in Core(\nu)} P(F_i) \\ &= \sum_{i=1}^{k} Q(F_i) \\ &= \min_{P \in Core(\nu)} \sum_{i=1}^{k} P(F_i).\end{aligned}$$

□

Definition 2. For a capacity ν with a non-empty core define the following functions $\hat{\nu} : \mathcal{C} \to \mathbb{R}$ and $\overline{\overline{\nu}} : \mathcal{C} \to \mathbb{R}$:

$$\hat{\nu}(F_1, \ldots, F_k) := \inf \left\{ \sum_{i=1}^{k} P(F_i) | P \in Core(\nu) \right\}.$$

$$\overline{\overline{\nu}}(F_1, \ldots, F_k) :=$$

$$\sup \left\{ \sum a_i \nu(E_i) - a | (a_i, E_i) \text{ finite sequence in } \mathbb{R}_+ \times \Sigma, a \in \mathbb{N}, \sum a_i E_i^* - aS^* \leq \sum_{i=1}^{k} F_i^* \right\}.$$

Lemma 4 implies that a capacity ν has a k - core if and only if $\sum_{i=1}^{k} \nu(F_i) = \hat{\nu}(F_1, \ldots, F_k)$ for all chains $(F_1, \ldots, F_k) \in \mathcal{F}_k$. The following Lemma establishes the remarkable observation that for capacities with a non-empty core $\hat{\nu}$ and $\overline{\overline{\nu}}$ are identical. Combined these two facts imply that for finite k, ν has a k - core if and only if $\sum_{i=1}^{k} \nu(F_i) = \overline{\overline{\nu}}(F_1, \ldots, F_k)$ for all chains $(F_1, \ldots, F_k) \in \mathcal{C}_k$. This result is crucial for the proof of our main theorem. The proof is inspired by Schmeidler (1972) where a similar approach is used.

Lemma 5. *Let ν be a capacity with a non-empty core. Then for all $(F_1, \ldots, F_k) \in \mathcal{C}$*

$$\hat{\nu}(F_1, \ldots, F_k) = \overline{\overline{\nu}}(F_1, \ldots, F_k).$$

Recall that k - core capacities have a non-empty core, therefore Lemma 5 holds for those.

To prove the Lemma a separating hyperplane theorem is used.

Theorem 1 (Separating Hyperplane Theorem (Dunford and Schwartz (1958)))**.** *In a linear topological space, any two disjoint convex sets, one of which has an interior point, can be separated by a non-zero continuous linear functional.*

Proof of Lemma 3. $\underline{\hat{\nu} \geq \overline{\overline{\nu}}}$: Consider a chain $(F_1, \ldots, F_k) \in \mathcal{C}$ and an arbitrary $P \in Core(\nu)$. Assume that

$$\sum a_i E_i^* - aS^* \leq F_1^* + \cdots + F_k^*.$$

It follows that

$$\begin{aligned} P(F_1) + \cdots + P(F_k) &= P^*(F_1^*) + \cdots + P^*(F_k^*) \\ &= P^*(F_1^* + \cdots + F_k^*) \\ &\geq P^*\left(\sum a_i E_i^* - aS^*\right) \\ &= \sum a_i P^*(E_i^*) - aP^*(S^*) \\ &= \sum a_i P(E_i) - a \\ &\geq \sum a_i \nu(E_i) - a. \end{aligned}$$

It follows that $P(F_1) + \cdots + P(F_k) \geq \overline{\overline{\nu}}(F_1, \ldots, F_k)$. Since P was arbitrary and $Core(\nu)$ is compact we have that $\hat{\nu}(F_1, \ldots, F_k) \geq \overline{\overline{\nu}}(F_1, \ldots, F_k)$.

$\underline{\hat{\nu} \leq \overline{\overline{\nu}}}$: Consider a chain $(F_1, \ldots, F_k) \in \mathcal{C}$. The idea is to construct a set function F which is in $core(\overline{\overline{\nu}})$ such that $F(F_1^* + \cdots + F_k^*) = \overline{\overline{\nu}}(F_1, \ldots, F_k)$.

Define $f : \mathbf{B}(\Sigma) \to \mathbb{R}$:

$$f(x) = \sup\left\{\sum a_i \nu(E_i) - a | (a_i, E_i) \text{ finite sequence in } \mathbb{R}_+ \times \Sigma, a \in \mathbb{N}, \sum a_i E_i^* - aS^* \leq x\right\}.$$

The function f extends $\overline{\overline{\nu}}$ to $B(\Sigma)$. It is obviously monotonic. Schmeidler (1972) shows that f is also homogenous and superlinear, i.e.

$$f(\alpha x) = \alpha f(x) \qquad \forall \alpha \in \mathbb{R}_+ \quad \forall x \in \mathbf{B}(\Sigma)$$
$$f(x+y) \geq f(x) + f(y) \qquad \forall x, y \in \mathbf{B}(\Sigma).$$

Consider $(F_1, \ldots, F_k) \in \mathcal{C}$.

Case 1: $f\left(\sum_{i=1}^{k} F_i^*\right) \neq 0$. Consider the following sets

$$A = \{x \in \mathbf{B}(\Sigma) | f(x) \geq 1\}, \qquad B = \{x \in \mathbf{B}(\Sigma) | x \leq S^*\},$$
$$C = \left\{ x \in \mathbf{B}(\Sigma) | x \leq \frac{\sum_{i=1}^{k} F_i^*}{f\left(\sum_{i=1}^{k} F_i^*\right)} \right\}.$$

The set A is convex due to the superlinearity of f and the set $D := Co(B \cup C)$ is convex by construction. $int(A)$ in convex and has an interior point. To apply Theorem 3 for the sets $int(A)$ and D we need to show that $int(A) \cap D = \emptyset$. It suffices to show that $f(x) \leq 1$ for all $x \in D$. Due to the monotonicity of f it suffices to show that $f\left(rS^* + (1-r)\frac{\sum_{i=1}^{k} F_i^*}{f\left(\sum_{i=1}^{k} F_i^*\right)}\right) \leq 1$ for $0 \leq r \leq 1$. Due to the homogeneity of f it suffices to show that

$$f\left(S^* + t\sum_{i=1}^{k} F_i^*\right) = 1 + tf\left(\sum_{i=1}^{k} F_i^*\right) \tag{2}$$

for $t > 0$. The $\geq$ direction follows from the superlinearity of f. For the $\leq$ direction let

$$\sum a_i E_i^* - aS^* \leq S^* + t\sum_{i=1}^{k} F_i^*.$$

Rearrange to get

$$\sum a_i E_i^* - (1+a)S^* \leq t\sum_{i=1}^{k} F_i^*.$$

From the definition of f we get

$$\sum a_i \nu(E_i) - (1+a) \leq tf\left(\sum_{i=1}^{k} F_i^*\right),$$

implying

$$\sum a_i \nu(E_i) - a \leq 1 + tf\left(\sum_{i=1}^{k} F_i^*\right).$$

It therefore holds that

$$f\left(S^* + t\sum_{i=1}^{k} F_i^*\right) \leq 1 + tf\left(\sum_{i=1}^{k} F_i^*\right).$$

The requirements for Theorem 2 for the sets $int(A)$ and D are therefore fulfilled and we can conclude the existence of a linear functional $F : \mathbf{B}(\Sigma) \to \mathbb{R}$ such that

$$F(x) \leq F(S^*) = 1 = F\left(\frac{\sum_{i=1}^{k} F_i^*}{f\left(\sum_{i=1}^{k} F_i^*\right)}\right) \leq F(y)$$

for all $x \in D$ and $y \in A$. The first equality is due to normalization, the second equality is due to the fact that both S^* and $\frac{\sum_{i=1}^{k} F_i^*}{f\left(\sum_{i=1}^{k} F_i^*\right)}$ are in $A \cap D$. Since F is linear and $F(S^*) = 1$ we have that $F_\Sigma \in \Delta^{|S|}$, where F_Σ denotes the function F restricted to Σ. The second equality and the linearity of F furthermore imply that $F\left(\sum_{i=1}^{k} F_i^*\right) = f\left(\sum_{i=1}^{k} F_i^*\right)$.

We need to show that $f(T^*) \leq F(T^*)$ for all $T \in \Sigma$, which implies that $F_\Sigma \geq \overline{\overline{\nu}}$ and so $F_\Sigma \in Core(\overline{\overline{\nu}})$.

Fix some $T \in \Sigma$. The linearity of F implies that $F(T^*) \geq 0$. If $f(T^*) = 0$ then $F(T^*) \geq f(T^*)$. So assume that $f(T^*) > 0$. There exists an $r > 0$ such that $f(rT^*) = 1$. This implies that $rT^* \in A$ and therefore $F(T^*) \geq 1$. Homogeneity of f and F then implies the required $f(T^*) \leq F(T^*)$. Therefore

$$\overline{\overline{\nu}}(F_1, \ldots, F_k) = F(F_1^* + \cdots + F_k^*) \geq \inf\{P(F_1) + \cdots + P(F_k) | P \in Core(\nu)\} = \hat{\nu}(F_1, \ldots, F_k).$$

Case 2: $f\left(\sum_{i=1}^{k} F_i^*\right) = 0$. Define A and B as above and

$$C = \bigcup_{n=1}^{\infty}\left\{x \in \mathbf{B}(\Sigma) | x \leq n\sum_{i=1}^{k} F_i^*\right\}.$$

To show $int(A) \cap D = \emptyset$ we show with the same approach as above that for $0 \leq r \leq 1$

$$f\left(rS^* + (1-r)n\sum_{i=1}^{k} F_i^*\right) = r + (1-r)nf\left(\sum_{i=1}^{k} F_i^*\right)$$
$$= r \leq 1 \qquad\qquad \forall 0 \leq r \leq 1,$$

So again we can use Theorem 3 to get the separating function F as above. We need to show $F\left(\sum_{i=1}^{k} F_i^*\right) = 0$. Assume that $F\left(\sum_{i=1}^{k} F_i^*\right) > 0$. Then there exists an n such that $F\left(n\sum_{i=1}^{k} F_i^*\right) > 1$. So $n\sum_{i=1}^{k} F_i^* \in A$ as well as $n\sum_{i=1}^{k} F_i^* \in D$ which contradicts the separating property of F. Therefore $F\left(\sum_{i=1}^{k} F_i^*\right) = 0$. To show $F(T^*) \geq f(T^*)$ for all $T \in \Sigma$ the above approach is used. So $F_\Sigma \in Core(\nu)$ which finishes the proof.

□

Consider the following slightly more general version of the Main Theorem.

Theorem 2. *Consider a preference relation $\succsim$ satisfying the standard CEU axioms with corresponding capacity ν. The following are equivalent:*

(i) ν has a k - core.

(ii) For all $(F_1, \ldots, F_k) \in \mathcal{C}_k$: $\sum_{i=1}^{k} \nu(F_i) = \overline{\overline{\nu}}(F_1, \ldots, F_k)$

(iii) $\succsim$ satisfies the axiom k - Ambiguity Aversion

Proof. "$(i) \iff (ii)$": Shown in Lemma 3.

"$(iii) \implies (ii)$": Assume that (ii) fails. Then there exists a chain $(F_1, \ldots, F_k) \in \mathcal{C}_k$ such that $\sum_{i=1}^{k} \nu(F_i) < \overline{\overline{\nu}}(F_1, \ldots, F_k)$. It follows that there exist $(a_i, E_i)_{i=1}^m \in \mathbb{R}_+^m \times \Sigma^m, a \in \mathbb{N}$ with

$$\sum_{i=1}^{m} a_i E_i^* - aS^* = \sum_{i=1}^{k} F_i^* \tag{3}$$

such that

$$\sum_{i=1}^{k} \nu(F_i) < \sum_{i=1}^{m} a_i \nu(E_i) - a. \tag{4}$$

We get strict equality in (3) since we can always add other terms of the form $a_i E_i^*$. Assume acts in utility terms?

Now define the acts $h = \sum_{i=1}^{k} 1_{F_i} 0$ and $f_i = a_{i E_i} 0$ for $i \in \{1, \dots m\}$ and $f_{m+1} = -a$. By (3) it holds that

$$h = \sum_{i=1}^{m+1} f_i.$$

By the comonotonic independence axiom and the fact that constant acts are comonotonic to all acts we can now find constant acts $b_1, \dots, b_{m+1} \in \mathcal{F}_1$ with $\sum_{i=1}^{k+1} b_i = 0$ such that for the acts $f'_i := f_i + b_i, i \in \{1, \dots, k+1\}$ it holds that

$$f'_1 \sim \dots \sim f'_{k+1}.$$

Obviously $h = \sum_{i=1}^{m+1} f'_i$. For $f^\dagger_i := (m+1) f'_i$ it holds that

$$f^\dagger_1 \sim \dots \sim f^\dagger_{m+1} \tag{5}$$

and also

$$h = \sum_{i=1}^{m+1} \frac{1}{m+1} f^\dagger_i. \tag{6}$$

Now

$$\begin{aligned}
\int h \, d\nu &= \sum_{i=1}^{k} \nu(F_i) \\
&\overset{(6)}{<} \sum_{i=1}^{m} a_i \nu(E_i) - a \\
&= \sum_{i=1}^{m+1} \int f_i \, d\nu \\
&= \sum_{i=1}^{m+1} \int f'_i \, d\nu \\
&= \sum_{i=1}^{m+1} \frac{1}{m+1} \int f^\dagger_i \, d\nu \\
&= \int f^\dagger_1 \, d\nu,
\end{aligned}$$

where the first equality holds since $(F_1, \dots, F_k)$ is a chain. Therefore $f^\dagger_1 \succ h$ and this in combination with (5) and (6) contradicts the axiom k - Ambiguity Aversion.

"$(i) \implies (iii)$": Consider some act $h \in \mathcal{F}_k$. Since ν has a k - core it follows that

$$\int h \, d\nu = \min_{P \in Core(\nu)} \int h \, dP.$$

Assume that we have binary acts $f_1 \sim \dots \sim f_m$ with $\sum_{i=1}^{m} \alpha_i f_i = h, \sum_{i=1}^{m} \alpha_i = 1$ and $\alpha_i \geq 0$.

Consider some $Q \in \arg \min_{P \in Core(\nu)} \int h \, dP$. It holds that

$$\begin{aligned}
\int f_1 \, d\nu &= \sum_{i=1}^{m} \alpha_i \int f_i \, d\nu \\
&= \sum_{i=1}^{m} \alpha_i \min_{P \in Core(\nu)} \int f_i \, dP \\
&\leq \sum_{i=1}^{m} \alpha_i \int f_i \, dQ \\
&= \int \sum_{i=1}^{m} \alpha_i f_i \, dQ \\
&= \int h \, dQ \\
&= \int h \, d\nu,
\end{aligned}$$

where the inequality holds since $Q \in Core(\nu)$. We have therefore shown that $h \succeq f_1$, so the axiom k - Ambiguity Aversion holds.

□

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